Conditions Equivalent to Seminormality in Certain Classes of Commutative Rings

DAVID F. ANDERSON

Department of Mathematics
The University of Tennessee
Knoxville, Tennessee 37996-1300
U. S. A.

AYMAN BADAWI

Department of Mathematics
Birzeit University, Box 14
Birzeit, WestBank via Israel

Abstract

In this paper, we consider several conditions on a commutative ring R which are related to seminormality. We show that they are all equivalent to seminormality if R is reduced with only a finite number of minimal prime ideals (in particular, if R is reduced and Noetherian).

Introduction

Let R be a commutative ring with 1 and total quotient ring $T(R)$. Recall that R is (2,3)-closed (or seminormal) if whenever $x^2, x^3 \in R$ for $x \in T(R)$, then $x \in R$. Based on earlier work of Traverso [12], Gilmer and Heitmann [4] showed that if R is either an integral domain or a reduced Noetherian ring, then $Pic(R[\mathbf{X}]) = Pic(R)$ if and only if R is (2,3)-closed (here $\mathbf{X}$ is any nonempty set of indeterminates, and recall that R is reduced if $nil(R) = \{0\}$, i.e., R has no nonzero nilpotent elements). An example in [4] shows that this result does not extend to arbitrary reduced commutative rings. Rush [10] extended this result to reduced commutative rings with only a finite number of minimal prime ideals.

Following Swan [11], we define a (reduced) commutative ring R to be seminormal if it satisfies the following condition:

If $a^2 = b^3$ for $a, b \in R$, then $a = k^3$ and $b = k^2$ for some $k \in R$.

Then $Pic(R[\mathbf{X}]) = Pic(R)$ if and only if $R_{red}(= R/nil(R))$ is seminormal [11, Theorem 1]. Note that a seminormal ring is (2,3)-closed, but a reduced (2,3)-closed ring need not be seminormal (see Section 3).

In this paper, we study several conditions on a commutative ring R which are related to seminormality. They are all equivalent if R is an integral domain, or more generally, if R is a reduced commutative ring with only a finite number of minimal prime ideals, equivalently, if R is a subring of a direct product of finitely many integral domains (in particular, if R is reduced and Noetherian); but they need not be equivalent for an arbitrary reduced commutative ring R. In the first section, we study these conditions for arbitrary commutative rings. In the second section, we specialize to (reduced) rings which are represented as subrings of direct products of integral domains. In the final section, we relate these conditions to (2,3)-closure and the Picard group. In particular, we show in Theorem 3.3 that $Pic(R[\mathbf{X}]) = Pic(R)$ for a reduced commutative ring R with only a finite number of minimal prime ideals if and only if whenever $a^2 = b^3$ for regular elements a and b of R, then $a = bk$ for some $k \in R$.

Throughout, all rings are commutative with 1. When we say that a ring A is a subring of a ring B, we mean that A and B have the same identity element. As usual, $x \in R$ is called a regular element if x is not a zerodivisor of R. Also, X will denote a single indeterminate and $\mathbf{X}$ a nonempty set of indeterminates. Any undefined terminology or notation is standard, as in [3, 5, 6].

1 Elementary Results

In this section, we consider the following five conditions related to seminormality for a commutative ring R with $a, b \in R$. Conditions (2) and (3) are the easiest to check since they only involve divisibility in R.

(1) If $a^2 = b^3$, then $a = k^3$ and $b = k^2$ for some $k \in R$ (i.e., R is seminormal).

(2) If $a^2 = b^3$, then $a = bk$ for some $k \in R$ (i.e., $b|a$).

(3) If $a^2 = b^3$, then $b^2 = ak$ for some $k \in R$ (i.e., $a|b^2$).

(4) If $a^2 = b^3$, then $b^2 = c$ and $a^4 = c^3$ for some $c \in R$ with $c = ak$ for some $k \in R$ (i.e., $a|c$).

(5) If $a^2 = b^3$, then $a^2 = ak^3$ and $b^2 = bk^2$ for some $k \in R$.

Examples of seminormal rings include any integrally closed domain (cf. Section3), any von Neumann regular ring, and any direct product of seminormal rings. We show in Theorem 2.6 that all five conditions are equivalent for R a reduced commutative ring with only a finite number of minimal prime ideals (in particular, for R a reduced commutative Noetherian ring). We first give several elementary observations and a theorem.

REMARK 1.1. (a) In [11], Swan included "reduced" in his definition of seminormal ring. However, Costa observed that the ring R is necessarily reduced if it satisfies condition (1). Thus the k in condition (1) is unique (cf. [11, Lemma 3.1]). Also, R is reduced if it satisfies condition (2). It is sufficient to show that if condition (2) holds and $a^2 = 0$, then $a = 0$. Let $b = 0$. Then $a^2 = 0 = b^3$ yields $a = bk$ for some $k \in R$. Hence $a = 0$. However, one can easily show that $R = \mathbb{Z}_4$ satisfies conditions (3), (4), and (5) (see part (e) below). More generally, conditions (3), (4), and (5) all hold if R is quasilocal with maximal ideal M such that $M^2 = 0$. Hence condition (3), (4), or (5) on R need not imply that R is reduced, and thus need not imply conditions (1) or (2).

(b) We have already observed that in condition (1), k is uniquely determined. In condition (2), k is determined only up to $ann(b)$. In conditions (3) and (4), k is determined only up to $ann(a)$; while in condition (5), k is determined only up to $ann(a) \cap ann(b)$.

(c) Note that conditions (1) - (5) are all equivalent and k is uniquely determined if a and b are restricted to regular elements of R.

(d) Let $\{R_\alpha\}$ be a family of commutative rings with 1. Then the direct product $\prod R_\alpha$ satisfies any of the above five conditions if and only if each R_α satisfies that condition.

(e) Let $R = \mathbb{Z}_n$, where $n = p_1^{m_1} \cdots p_r^{m_r}$ for distinct primes p_i and integers $m_i \geq 1$. Then R satisfies conditions (1) or (2) $\Leftrightarrow$ each $m_i = 1$; and R satisfies conditions (3), (4), or (5) $\Leftrightarrow$ each $m_i \in \{1, 2, 4\}$. We leave the details to the reader.

THEOREM 1.2. *Let R be a direct product of integral domains. Then conditions (1) - (5) are all equivalent. Moreover, the same $k \in R$ works in each implication. In particular, they are all equivalent if R is an integral domain.*

Proof. If R is an integral domain and $0 \neq a, b \in R$, then the same $k(= a/b = b^2/a \in R)$ works for each condition. The general result is clear since the conditions all hold coordinatewise. □

We next give the possible implications between the five conditions. We have been unable to determine the relationship between conditions (3) and (5). Otherwise, examples given throughout the paper show that these are the only implications that hold in general. The following diagram summarizes the implications between conditions (1) - (5) in Theorem 1.3.

$$\begin{array}{ccccc} (1) & \Longrightarrow & (2) & \Longrightarrow & (5) \\ \Downarrow & & \Downarrow & & \\ (4) & \Longleftrightarrow & (3) & & \end{array}$$

THEOREM 1.3. *Let R be a commutative ring. Then (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), (1) $\Rightarrow$ (5), (2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), (2) $\Rightarrow$ (5), and (3) $\Leftrightarrow$ (4). Moreover, the same $k \in R$ works in each implication.*

Proof. The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), (1) $\Rightarrow$ (5), and (3) $\Leftrightarrow$ (4) are all easy and thus left to the reader. We do (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5). Thus also (2) $\Rightarrow$ (4) since (3) $\Rightarrow$ (4). The proof of each implication yields the "moreover" statement.

(2) $\Rightarrow$ (3): Suppose that (2) holds. Then R is reduced by Remark 1.1(a). Let $a, b \in R$ with $a^2 = b^3$. Then $a = bk$ for some $k \in R$. We show that $b^2 - ak \in P$

for each prime ideal P of R. If $b \in P$, then also $a \in P$, and hence $b^2 - ak \in P$. If $b \notin P$, then $b(b^2 - ak) = b^3 - a(bk) = a^2 - a^2 = 0 \in P$ yields $b^2 - ak \in P$. Thus $b^2 - ak = 0$ since R is reduced.

(2) $\Rightarrow$ (5): Suppose that (2) holds. Let $a, b \in R$ with $a^2 = b^3$. Then $a = bk$ for some $k \in R$, and by the proof of (2) $\Rightarrow$ (3) above, also $b^2 = ak$. Thus $b^2 = ak = (bk)k = bk^2$. Hence $a^2 = (bk)^2 = b^2k^2 = (ak)k^2 = ak^3$. □

In Remark 1.1(a), we observed that condition (3), (4), or (5) need not imply condition (1) or (2). However, we next show that conditions (2) - (5) are all equivalent if R is reduced.

THEOREM 1.4. *Let R be a reduced commutative ring. Then conditions (2), (3), (4), and (5) are all equivalent, and (1) $\Rightarrow$ (2). Moreover, the same $k \in R$ works in each implication.*

Proof. By Theorem 1.3, we need only show that (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2). The proof of each implication yields the "moreover" statement.

(3) $\Rightarrow$ (2): Suppose that (3) holds. Let $a, b \in R$ with $a^2 = b^3$. Then $b^2 = ak$ for some $k \in R$. Let P be a prime ideal of R. If $a \in P$, then also $b \in P$, and hence $a - bk \in P$. If $a \notin P$, then $a(a - bk) = a^2 - a(bk) = b^3 - b(ak) = b^3 - b^3 = 0 \in P$ yields $a - bk \in P$. Thus $a - bk = 0$ since R is reduced.

(5) $\Rightarrow$ (2): Suppose that (5) holds. Let $a, b \in R$ with $a^2 = b^3$. Then $a^2 = ak^3$ and $b^2 = bk^2$ for some $k \in R$. Let P be a prime ideal of R. If $a \in P$, then also $b \in P$, and hence $a - bk \in P$. If $a \notin P$ (thus $b \notin P$ and $ab \notin P$), then $ab(a - bk) = a^2b - ab^2k = (ak^3)b - a(bk^2)k = abk^3 - abk^3 = 0 \in P$ yields $a - bk \in P$. Hence $a - bk = 0$ since R is reduced. □

In Example 2.7 of the next section, we will give an example of a reduced commutative ring R for which (2) $\not\Rightarrow$ (1). However, Theorem 2.6 shows that all five conditions are equivalent if R is a reduced commutative ring with only a finite number of minimal prime ideals. The difficulty with the implication (2) $\Rightarrow$ (1) is that the same $k \in R$ which works in conditions (2) - (5) needs not work in condition (1) (cf. Remark 1.1(b)).

There are also other natural variants of condition (5). For each integer $n \geq 1$, consider the following condition on R:

(5_n) If $a^2 = b^3$, then $a^{n+1} = a^nk^3$ and $b^{n+1} = b^nk^2$ for some $k \in R$.

Then (5_n) $\Rightarrow$ (5_{n+1}) for each integer $n \geq 1$, and one may easily verify that (3) $\Rightarrow$ (5_n) for each integer $n \geq 3$, and $R = \mathbb{Z}_8$ satisfies condition (5_3), but neither condition (3) nor condition (5). A slight modification to the proof of (5) $\Rightarrow$ (2) of Theorem 1.4 shows that (5_n) $\Rightarrow$ (2) for each integer $n \geq 1$ when R is reduced.

2 Subrings of Direct Products of Integral Domains

In this section, we specialize to the case where R is a subring of a direct product of integral domains. Recall that a commutative ring R is reduced if and only if it is a subring of a direct product of integral domains (if $\cap P_\alpha = 0$ for some family of prime ideals $\{P_\alpha\}$ of R, then $R \subset \prod R/P_\alpha$). However, for constructing examples it is often easier to view R as a subring of a given $\prod R_\alpha$ rather than to embed a given reduced ring R in $\prod R/P_\alpha$.

Let $\{R_\alpha\}$ be a family of integral domains and let $R \subset \prod R_\alpha$. For $x = (x_\alpha) \in R$, let $supp(x) = \{\alpha | x_\alpha \neq 0\}$. For $x = (x_\alpha), y = (y_\alpha) \in R$, we say that x and y have the same type, denoted by $x \sim y$, if $x_\alpha = 0 \Leftrightarrow y_\alpha = 0$, i.e., $\Leftrightarrow supp(x) = supp(y)$. Note that $\sim$ is an equivalence relation and $x \sim x^n$ for all integers $n \geq 1$. Moreover, if $a \sim b$ and $c \sim d$ for $a, b, c, d \in R$, then $ac \sim bd$. Also, if $x \sim y$, then x is a regular element of $R \Leftrightarrow y$ is a regular element of R.

Note that the type of an element of R depends on how R is embedded as a subring in a direct product of integral domains. So when we talk about "type", we always assume a fixed embedding $R \subset \prod R_\alpha$. Let $\{P_\alpha\}$ be a family of prime ideals of R with $\cap P_\alpha = 0$; so $R \subset \prod R_\alpha/P_\alpha$. With respect to this embedding, then $(x_\alpha) \sim (y_\alpha)$ just means that $x_\alpha \in P_\alpha \Leftrightarrow y_\alpha \in P_\alpha$ for each α. Thus the techniques used in Sections 1 and 2 are really the same. In fact, we invite the reader to translate the proofs of the results in this section to coordinate-free proofs like those used in Section 1.

We say that $y = (y_\alpha) \in R \subset \prod R_\alpha$ is an extension of (or extends) $x = (x_\alpha) \in R$, denoted by yEx, if $y_\alpha = x_\alpha$ whenever $x_\alpha \neq 0$. Note that if x is a regular element of R and yEx, then y is also a regular element of R.

LEMMA 2.1. *Let R be a subring of a direct product of integral domains and $a, b, c \in R$ with $a \sim b \sim c$. If $ab = ac$, then $b = c$.*

Proof. Let $a = (a_\alpha), b = (b_\alpha)$, and $c = (c_\alpha)$. If $a_\alpha \neq 0$, then $b_\alpha = c_\alpha$ since R_α is an integral domain. If $a_\alpha = 0$, then $b_\alpha = c_\alpha = 0$ since $a \sim b \sim c$. Thus each $b_\alpha = c_\alpha$; so $b = c$. □

Theorem 1.4 of the previous section also gives the implications between the five conditions when R is a subring of a direct product of integral domains. For subrings of direct products of integral domains, these implications may also be proved using Lemma 2.1.

Example 2.7 shows that in general (2) $\not\Rightarrow$ (1). We next show, in the context of subrings of direct products of integral domains, that a variant of condition (2) is equivalent to condition (1). We leave the formulation for the arbitrary reduced ring case to the reader.

THEOREM 2.2. *Let R be a subring of a direct product of integral domains. Then the following two conditions on R are equivalent.*

(1) If $a^2 = b^3$ for $a, b \in R$, then $a = k^3$ and $b = k^2$ for some $k \in R$.

(2′) If $a^2 = b^3$ for $a, b \in R$, then $a = bk$ for some $k \in R$ with $b \sim k$.

Proof. (1) $\Rightarrow$ (2′): Suppose that (1) holds. Let $a^2 = b^3$. Then $a = k^3$ and $b = k^2$ for some $k \in R$. Thus $a = k^3 = k^2k = bk$ with $b = k^2 \sim k$.

(2′) $\Rightarrow$ (1): Suppose that (2′) holds. Let $a^2 = b^3$. Then $a = bk$ for some $k \in R$ with $b^2 \sim b \sim k \sim k^2$. Thus $b^2b = b^3 = a^2 = b^2k^2$ yields $b = k^2$ by Lemma 2.1, and hence also $a = bk = k^3$. □

We next show that another variant of condition (2) implies condition (1).

THEOREM 2.3. *Let R be a subring of a direct product of integral domains such that if $a^2 = b^3$ for $a, b \in R$, then $a = bc$ for some regular element $c \in R$. Then R satisfies condition (1).*

Proof. Let $a = (a_\alpha), b = (b_\alpha) \in R$ such that $a^2 = b^3$. Then $a = bc$ for some regular element $c = (c_\alpha) \in R$. Let $F = supp(a) = \{\alpha | a_\alpha \neq 0\}$. Then $b_\alpha \neq 0$ and $c_\alpha \neq 0$ for each $\alpha \in F$, and $b_\alpha = 0$ if $\alpha \notin F$. Since $a^2 = b^3$ and $a = bc$, we have $a_\alpha = c_\alpha^3$ and $b_\alpha = c_\alpha^2$ in the integral domain R_α for each $\alpha \in F$. Let $z = c^2 + b^2 - b$ and $w = c^3 + a^2 - a$. Then $z_\alpha = c_\alpha^4, w_\alpha = c_\alpha^6$ if $\alpha \in F$, and $z_\alpha = c_\alpha^2, w_\alpha = c_\alpha^3$ if $\alpha \notin F$. Thus z and w are both regular elements of R since c is regular and $w \sim c \sim z$. Since $w^2 = z^3$, by hypothesis $w = zd$ for some regular element $d = (d_\alpha) \in R$. Hence $w = d^3$ and $z = d^2$ by Remark 1.1(c). Thus $w \sim c \sim z \sim d$. Note that $d_\alpha = c_\alpha^2 = b_\alpha$ if $\alpha \in F$ and $d_\alpha = c_\alpha$ if $\alpha \notin F$. Let $k = c - (d - b) = (k_\alpha) \in R$. Then $k_\alpha = c_\alpha$ if $\alpha \in F$ and $k_\alpha = 0$ if $\alpha \notin F$. Hence $a = k^3$ and $b = k^2$; so R satisfies condition (1). (This also follows from Theorem 2.2 since $a = bk$ with $b \sim k$.) □

The next condition we consider is just condition (2) restricted to regular elements of R. (Note that b and k are also regular elements of R.)

(6) If $a^2 = b^3$ for $a, b \in R$ with b a regular element of R, then $a = bk$ for some $k \in R$ (i.e., $b|a$).

Clearly (2) $\Rightarrow$ (6). In fact, each of the five earlier conditions implies condition (6) since, as observed in Remark 1.1(c), conditions (1) - (5) are all equivalent when restricted to regular elements of R. Note that condition (6) holds for any commutative ring R with $T(R) = R$. In particular, $R = \mathbb{Z}_8$ satisfies condition (6), but none of conditions (1) - (5) (cf. Remark 1.1(e)). Also, the direct product $\prod R_\alpha$ satsifies condition (6) if and only if each ring R_α satisfies condition (6).

We next show that with an additional hypothesis, the conditions (1) - (6) are all equivalent.

THEOREM 2.4. *Let R be a subring of a direct product of integral domains such that if $a^2 = b^3$ for $a, b \in R$, then b can be extended to a regular element of R. Then conditions (1) - (6) are all equivalent.*

Proof. By Theorem 1.4, conditions (2) - (5) are all equivalent, and (1) $\Rightarrow$ (2). Since (2) $\Rightarrow$ (6), we need only show that (6) $\Rightarrow$ (1). Suppose that (6) holds and let $a^2 = b^3$ for $a = (a_\alpha), b = (b_\alpha) \in R \subset \prod R_\alpha$. By hypothesis, there is a regular element $c = (c_\alpha)$ of R such that cEb. Let $z = c - b$, and let $x = z^2 + b$ and $y = z^3 + a$. Let $F = supp(b)$. Then $x_\alpha = b_\alpha = c_\alpha, y_\alpha = a_\alpha$ if $\alpha \in F$; and $x_\alpha = c_\alpha^2, y_\alpha = c_\alpha^3$ if $\alpha \notin F$. Thus x and y are regular elements of R since $x \sim c \sim y$. Also, yEa, xEb, and $y^2 = x^3$. By (6), $y = xd$ for some regular element d of R. Thus $a = bd$ since yEa, xEb, and $y = xd$. Hence R satisfies (1) by Theorem 2.3. □

We next investigate the case where R is a subring of a direct product of finitely many integral domains. Note that a commutative ring R is a subring of a direct product of finitely many integral domains if and only if R is reduced with only a finite number of minimal prime ideals (if $R \subset R_1 \times \cdots \times R_n$, then a minimal prime ideal of R has the form $(R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n) \cap R$ for some $1 \leq i \leq n$). First a key lemma; Example 2.7 shows that Lemma 2.5 does not extend to a direct product of infinitely many integral domains.

LEMMA 2.5. *Let R be a subring of a direct product of finitely many integral domains. Then each $x \in R$ may be extended to a regular element of R.*

Proof. We may assume that x is a zerodivisor of R. Thus $xy = 0$ for some $0 \neq y \in R$. Let $z = x + y$. Then zEx and $supp(x) \subsetneq supp(z)$. Continuing in this manner, after finitely many steps we reach a regular element of R which extends x. □

THEOREM 2.6. *Let R be a subring of a direct product of finitely many integral domains. Then conditions (1) - (6) are all equivalent. In particular, they are all equivalent if R is a reduced commutative ring with only a finite number of minimal prime ideals (for example, if R is a reduced commutative Noetherian ring).*

Proof. Since each element of R may be extended to a regular element of R by Lemma 2.5, the first part of this theorem is a direct consequence of Theorem 2.4. For the "in particular" statement, just note that a commutative ring may be embedded in a direct product of finitely many integral domains if and only if it is reduced and has only a finite number of minimal prime ideals, and that a commutative Noetherian ring has only a finite number of minimal prime ideals [6, Theorem 88]. □

The next example shows that conditions (1) and (2) are not equivalent for subrings of a direct product of infinitely many integral domains, i.e., for arbitrary reduced commutative rings.

EXAMPLE 2.7. (a) Let R be the subring of $T = \prod_{n\geq 1} \mathbb{Z}_2[X]$ generated by $(1,1,1,\dots)$ and $\{y_n = X(e_n + e_{n+1}) | n \geq 1\}$ (here $e_m \in T$ has a 1 in the mth slot and 0 elsewhere). Then $R = \mathbb{Z}_2(1,1,1,\dots) + \{(f_n) \in \bigoplus X\mathbb{Z}_2[X] \mid |\{f_n | ord\, f_n = 1\}|$ is even $\}$. Let $a = (X^3, 0, \dots), b = (X^2, 0, \dots) \in R$. Then $a^2 = b^3$, but there is no $k \in R$ with $a = k^3$ and $b = k^2$. Thus R is a reduced commutative ring for which condition (1) fails. Note that $a = y_1 b$ and $b \in R$ cannot be extended to a regular element of R.

We next show that R satisfies condition (2). Suppose that $a^2 = b^3$ for $a = (a_n), b = (b_n) \in R$. Then each $a_n = b_n k_n$ for some $k_n \in \mathbb{Z}_2[X]$ since $\mathbb{Z}_2[X]$ is integrally closed, and hence seminormal. If a_n is "eventually" 0, then we may assume that k_n is "eventually" 0. So we may assume that $(k_n) \in R$ (add some Xe_m if necessary). If a_n is "eventually" 1, then each $a_n(0) = b_n(0) = k_n(0) = 1$. Since $a_n^2 = b_n^3$ and $char R = 2$, then b_n has no X term. Thus k_n has an X term if and only if a_n has an X term; so $k = (k_n) \in R$. Hence $a = bk$; so condition (2) holds.

If we modify the above example by using $\mathbb{Z}_2[\![X]\!]$ rather than $\mathbb{Z}_2[X]$, then $R = \mathbb{Z}_2(1,1,1,\dots) + \{(f_n) \in \bigoplus X\mathbb{Z}_2[\![X]\!] \mid |\{f_n | ordf_n = 1\}|$ is even $\} \subset \prod_{n\geq 1} \mathbb{Z}_2[\![X]\!]$ satisfies condition (2), but not condition (1), and each element of R is either a unit or a zerodivisor of R; so $T(R) = R$.

(b) Let K be any field and $R = K(1,1,1,\dots) + \bigoplus X^2 K[\![X]\!] + \{\alpha_1 y_1 + \cdots + \alpha_n y_n | \alpha_i \in K$ and $n \geq 1\} \subset \prod_{n\geq 1} K[\![X]\!]$, where $y_n = X(e_n + e_{n+1})$ as in part (a) above. Then one may easily verify that R satisfies condition (2) and $T(R) = R$, but R does not satisfy condition (1).

3 (2,3)-Closure, Seminormality, and the Picard Group

In this section, we relate the six conditions from Sections 1 and 2 to (2,3)-closure and when $Pic(R[\mathbf{X}]) = Pic(R)$. We first state three more conditions for a commutative ring R with integral closure $\overline{R}$.

(7) If $x^2, x^3 \in R$ for $x \in T(R)$, then $x \in R$ (i.e., R is (2,3)-closed).

(8) If $x^2, x^3 \in R$ for x a unit of $T(R)$, then $x \in R$.

(9) $R = R^+ := \{x \in \overline{R} | x/1 \in R_p + J(\overline{R}_p)$ for every $p \in Spec(\overline{R})\}$, where $J(A)$ denotes the Jacobson radical of the ring A.

The original definition of "seminormal" in [12] for a commutative Noetherian ring R with finite integral closure $\overline{R}$ was that $R = R^+$. The equivalence of conditions (7) and (9) (with no finiteness assumption on $\overline{R}$) was shown in [4, Theorem 1.1]. Note that condition (7) holds by default for any commutative ring R with $T(R) = R$, and thus condition (7) does not imply that R is reduced. Hence (7) $\not\Rightarrow$ (1) in general. Examples in [3] and [11], and Examples 2.7 and 3.5, show that (7) $\not\Rightarrow$ (1) even for reduced rings R with $T(R) = R$. Clearly (7) $\Rightarrow$ (8). Although some evidence indicates that (8) $\not\Rightarrow$ (7), we have not been able to find such an example. For a survey of (2,3)-closure and root closure, see [1].

We next determine the precise relationship between conditions (1) and (7), and show that conditions (6) and (8) are equivalent.

THEOREM 3.1. *Let R be a commutative ring. Then*

(a) R satisfies condition (1) if and only if R satisfies condition (7) and $T(R)$ satisfies condition (1). In particular, (1) $\Rightarrow$ (7).

(b) (7) $\Rightarrow$ (6).

(c) (6) $\Leftrightarrow$ (8).

Proof. (a) First suppose that R satisfies (1). Let S be the set of regular elements of R. Then $T(R) = R_S$ also satisfies (1) by [11, Proposition 3.7]. Let $x \in T(R)$ with $x^2, x^3 \in R$. Then $(x^3)^2 = (x^2)^3$, and hence $x^3 = k^3$ and $x^2 = k^2$ for some $k \in R$ since R satisfies (1). By [11, Lemma 3.1], then $x = k \in R$ since R, and hence $T(R)$, is reduced. Thus R satisfies (7).

Conversely, suppose that R satisfies (7) and $T(R)$ satisfies (1). Let $a, b \in R$ with $a^2 = b^3$. Then $a = k^3$ and $b = k^2$ for some $k \in T(R)$ since $T(R)$ satisfies (1). Thus $k^2, k^3 \in R$ implies $k \in R$ by (7). Hence R satisfies (1).

(b) Suppose that R satisfies (7). Let $a^2 = b^3$ with b a regular element of R. Let $x = a/b \in T(R)$. Then $x^2 = a^2/b^2 = b \in R$ and $x^3 = a^3/b^3 = a \in R$. Thus $a/b \in R$; so $b|a$. Hence R satisfies (6).

(c) Suppose that R satisfies condition (6) and that $x^2, x^3 \in R$ for some unit x of $T(R)$. Then $x^2 = c$ and $x^3 = d$ for regular elements c and d of R. Hence $x^6 = c^3 = d^2$. Thus $c|d$ by hypothesis; so $d = ck$ for k a regular element of R. Hence $xc = x^3 = d = ck$, and thus $x = k \in R$. Hence R satisfies condition (8). The proof of (8) $\Rightarrow$ (6) is similar to the proof of part (b) above since a/b is a unit of $T(R)$ if a and b are regular elements of R. □

COROLLARY 3.2. *Let R be a reduced commutative ring with only a finite number of minimal prime ideals. Then conditions (1) - (9) are all equivalent. In particular, they are all equivalent if R is a reduced commutative Noetherian ring.*

Proof. In this case, $T(R)$ is a direct product of finitely many fields [11, Corollary 3.6], and thus $T(R)$ satisfies condition (1). Hence conditions (1) - (8) are equivalent by Theorem 2.6 and Theorem 3.1. Conditions (7) and (9) are equivalent by [4, Theorem 1.1]. □

As mentioned in the Introduction, a commutative ring R satisfies $Pic(R[\mathbf{X}]) = Pic(R)$ if and only if R_{red} is seminormal (i.e., satisfies condition (1)); thus if and only if R_{red} is (2,3)-closed and $T(R_{red})$ is seminormal. In particular, if R has only a finite number of minimal prime ideals, then $Pic(R[\mathbf{X}]) = Pic(R)$ if and only if R_{red} satisfies any of conditions (1) - (9). Conditions (2), (3), and (6) are the easiest to check since they only involve divisibility in R_{red}. Also, if $Pic(R[\mathbf{X}]) = Pic(R)$ for a reduced commutative ring R, then R is seminormal, and hence R is (2,3)-closed; this has been observed in [4, Theorem 1.5] and [10, Theorem 3].

We next isolate, in the context of subrings of direct products of integral domains, the most important equivalent conditions to have $Pic(R[\mathbf{X}]) = Pic(R)$. Example 2.7 shows that the hypothesis that b can be extended to a regular element of R is needed.

THEOREM 3.3. *Let R be a subring of a direct product of integral domains such that if $a^2 = b^3$ for $a, b \in R$, then b can be extended to a regular element of R. Then the following four statements are equivalent.*

(a) $Pic(R[\mathbf{X}]) = Pic(R)$.

(b) *R is seminormal (i.e., R satisfies condition (1)).*

(c) *If $x^2, x^3 \in R$ for some $x \in T(R)$, then $x \in R$ (i.e., R satisfies condition (7)).*

(d) *If $a^2 = b^3$ for some $a, b \in R$ with b a regular element of R, then $b|a$ (i.e., R satisfies condition (6)).*

In particular, all four statements are equivalent if R is a subring of a direct product of finitely many integral domains. Thus all four statements are equivalent if R is a reduced commutative ring with only a finite number of minimal prime ideals (for example, if R is a reduced commutative Noetherian ring).

Proof. We have already observed that (a) and (b) are equivalent by [11, Theorem 1], and (b) and (d) are equivalent by Theorem 2.4. By Theorem 3.1, (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d). □

We end this paper by considering which of the conditions are preserved by adjoining indeterminates. By the Picard group characterization of seminormality, $R[\mathbf{X}]$ is seminormal (i.e., satisfies condition (1)) $\Leftrightarrow$ R is seminormal. However, $R = \mathbb{Z}_4$ satisfies conditions (3), (4), (5), and (6), while $\mathbb{Z}_4[X]$ does not. To see this, note that $a = X^3 + 2$ and $b = X^2$ are regular elements of $\mathbb{Z}_4[X]$ which satisfy $a^2 = X^6 = b^3$, but there does not exist a $k \in \mathbb{Z}_4[X]$ which satisfies any of conditions (3) - (6).

It is of more interest to determine when $R[\mathbf{X}]$ is (2,3)-closed. First note that if $R[\mathbf{X}]$ is (2,3)-closed, then R is reduced and (2,3)-closed. However, even if R is reduced, then R (2,3)-closed need not imply that $R[\mathbf{X}]$ is (2,3)-closed (see [2, Example 1]). In [2, Theorem 2], Brewer, Costa, and McCrimmon showed that $R[X]$ is (2,3)-closed if R is (2,3)-closed and $T(R)$ is von Neumann regular. We can slightly sharpen their result to just assuming that $T(R)$ is seminormal. (Recall that a von Neumann regular ring R is seminormal with $T(R) = R$, but a seminormal ring R with $T(R) = R$ need not be von Neumann regular. For example, let K be any field; then $R = K(1,1,1,\ldots) + \bigoplus XK[\![X]\!] \subset \prod_{n\geq 1} K[\![X]\!]$ is seminormal with $T(R) = R$, but R is not von Neumann regular.)

THEOREM 3.4. *Let R be a (reduced) commutative ring such that $T(R)$ is seminormal. Then R is (2,3)-closed if and only if $R[\mathbf{X}]$ is (2,3)-closed.*

Proof. If $R[\mathbf{X}]$ is (2,3)-closed, then certainly R is also (2,3)-closed. Conversely, suppose that R is (2,3)-closed and $T(R)$ is seminormal. Then R is seminormal (i.e., satisfies condition (1)) by Theorem 3.1(a). Hence $R[\mathbf{X}]$ is seminormal, and thus $R[\mathbf{X}]$ is (2,3)-closed by Theorem 3.1(a). □

As observed above, if R is seminormal, then $R[\mathbf{X}]$ is seminormal, and hence $R[\mathbf{X}]$ is also (2,3)-closed. However, our next example shows that it is possible to have $R[\mathbf{X}]$ (2,3)-closed, in fact, integrally closed, but R is not seminormal.

EXAMPLE 3.5. Let K be any field, Y an indeterminate, and $R = K(1,1,1,\ldots) + \bigoplus Y^2K[\![Y]\!] \subset \prod_{n\geq 1} K[\![Y]\!]$. Then clearly $T(R) = R$; so R is integrally closed, but R is not seminormal. By [8, Corollary 7], $R[X]$ is integrally closed, and hence (2,3)-closed. Thus $R[X]$ (2,3)-closed does not imply that R is seminormal. In fact, R is integrally closed and satisfies each of the conditions (6) - (9), but R does not satisfy any of conditions (1) - (5).

Finally, note that for a reduced commutative ring R, Lucas [9, Corollary 6] has shown that $R[X]$ is (2,3)-closed if and only if R is (2,3)-closed in $Q_o(R)$, where $Q_o(R)$ is the ring of finite fractions of R (cf. [7, pp. 36-46]).

Acknowledgements

The second author would like to thank the University of Tennessee for a fantastic time while visiting there during 1998-99.

References

[1] D. F. Anderson, Root closure in commutative rings, a survey, Lecture Notes in Pure and Applied Mathematics, vol. 205, Marcel Dekker, New York, 1999, 55-71.

[2] J. W. Brewer, D. L. Costa, and K. McCrimmon, Seminormality and root closure in polynomial rings and algebraic curves, J. Algebra 58(1979), 217-226.

[3] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[4] R. Gilmer and R. Heitmann, On $Pic(R[X])$ for R seminormal, J. Pure Appl. Algebra 16(1980), 251-257.

[5] J. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York, 1988.

[6] I. Kaplansky, *Commutative Rings*, Univ. of Chicago Press, Chicago, rev. ed., 1974.

[7] J. Lambeck, *Lectures on Rings and Modules*, Chelsea, New York, 1986.

[8] T. G. Lucas, Characterizing when $R[X]$ is integrally closed, Proc. Amer. Math. Soc. 105(1989), 861-867.

[9] T. G. Lucas, Characterizing when $R[X]$ is integrally closed, II, J. Pure Appl. Algebra 61(1989), 49-52.

[10] D. E. Rush, Seminormality, J. Algebra 67(1980), 377-384.

[11] R. G. Swan, On seminormality, J. Algebra 67(1980), 210-229.

[12] C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa 24(1970), 585-595.